MIAMI-DADE COUNTY PUBLIC SCHOOLS

SYNERGY

Florida Standards Instructional Planning Academy

DIVISION OF ACADEMIC SUPPORT

K-8 Mathematics Hand-Outs

Teachers

Using Self Sustaining Collaborative Structures to Increase Student Achievement

NGSSS vs. M AFS

Problems

3rd Grade – NGSSS – MA.3.A.6.1

Ms. Tanaka is ordering calendars for the students at 4 elementary schools. The table below shows the number of students at each of the schools.

Which is the **best** estimate of the total numbers of calendars Ms. Tanaka needs to order for all 4 schools?

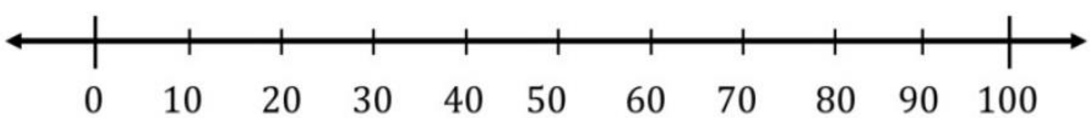
STUDENTS IN ELEMENTARY SCHOOLS

Name of School	Number of Students
Greendale	1,789
Jones Park	1,032
Shady River	2,115
Wakefield	1,992

A. 4,000
B. 5,000
★C. 7,000
D. 8,000

3rd Grade – MAFS.3.NBT.1.1

Plot 8, 32, and 79 on the number line.



A. Round each number to the nearest 10. How can you see this on the number line?
B. Round each number to the nearest 100. How can you see this on the number line?

NGSSS vs. MAFS
Problems

4th Grade – NGSSS – MA.4.G.3.2

Mr. Clark hired workers to construct an in-ground pool in his backyard. For which of the following situations might the workers have used the area formula when constructing the pool?

A. Determining the amount of water needed to fill the pool
B. Determining the amount of fencing to put around the pool
★C. Determining the amount of ground the bottom of the pool will cover
D. Determining the amount of dirt that will need to be dug for the pool

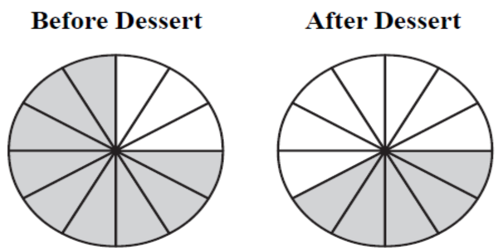
4th Grade – MAFS.4.MD.1.3

Karl’s rectangular vegetable garden is 20 feet by 45 feet, and Makenna’s is 25 feet by 40 feet. Whose garden is larger in area?

NGSSS vs. MAFS
Problems

5th Grade – NGSSS – MA.5.A.2.2

Mrs. Bradford served part of a pie for dessert. The shaded parts of the picture below show how much of the pie was in the pie plate before and after dessert.



What fraction of the whole pie, expressed in lowest terms, was eaten for dessert?

★ **A.** $\frac{1}{3}$

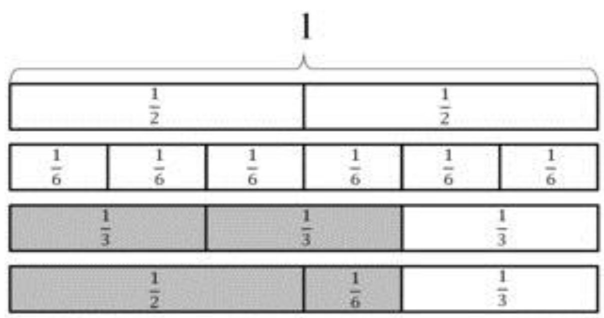
B. $\frac{5}{12}$

C. $\frac{7}{12}$

D. $\frac{3}{4}$

5th Grade – MAFS.5.NF.1.1

Ancient Egytians used unit fractions such as $\frac{1}{2}$ and $\frac{1}{3}$, to represent all fractions. For example, they might write the number $\frac{2}{3}$ as $\frac{1}{2}+\frac{1}{6}$.



We often think of $\frac{2}{3}$ as $\frac{1}{3}+\frac{1}{3}$, but the ancient Egyptians would not write it this way because they didn't use the same unit fraction twice.

a. Write each of the following Egyptian fractions as a single fraction:
   i. $\frac{1}{2}+\frac{1}{3}$ ii. $\frac{1}{2}+\frac{1}{3}+\frac{1}{15}$ iii. $\frac{1}{4}+\frac{1}{5}+\frac{1}{12}$

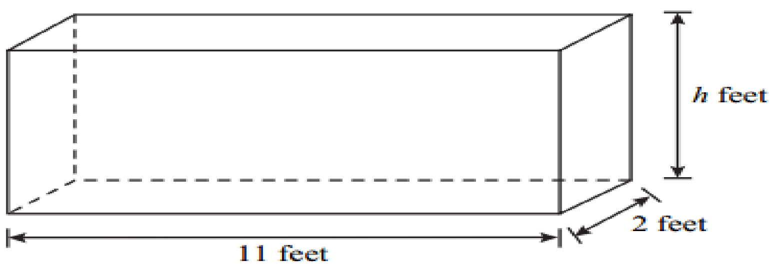
b. How might the ancient Egyptians have written the fraction we write as $\frac{3}{4}$?

NGSSS vs. MAFS
Problems

6th Grade – NGSSS – MA.6.G.4.3

Lamar is building a glass case for a reptile display. The interior of the case is in the shape of a rectangular prism with the dimensions shown in the diagram.

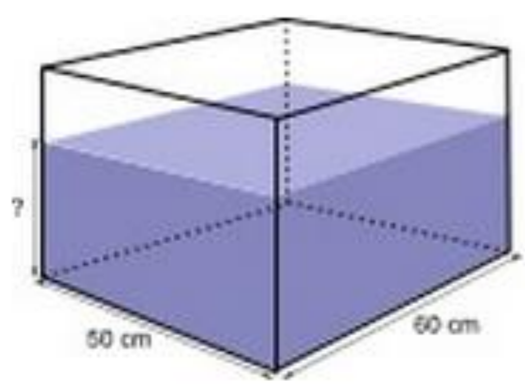
The total volume of the interior of the case is 154 cubic feet. What is the value of h?



★A. 7
B. 9
C. 13
D. 14

6th Grade – MAFS.6.G.1.2

A rectangular tank is **50*cm*** wide and **60*cm*** long. It can hold up **to 126 ℓ** of water when full. If Amy fills $\frac{2}{3}$ of the tank as shown, find the height of the water in centimeters. (Recall that 1 ℓ = $1000cm^3$.)



NGSSS vs. MAFS
Problems

7th Grade – NGSSS – MA.7.G.4.1

Toni has a rectangular vegetable garden that measures 12 feet by 18 feet. She wants to reduce the area of her garden. If Toni reduces the dimensions of her garden to 12 feet by 9 feet, how will the area of the new garden compare to the area of the old garden?

★ A. The area will be one-half as large.
B. The area will be two-thirds as large.
C. The area will be one-fourth as large.
D. The area will be three-fourths as large.

7th Grade – MAFS.7.G.2.6

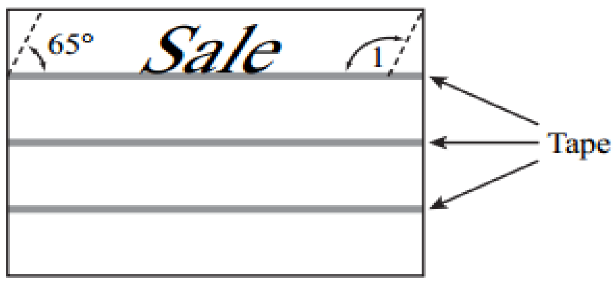
The 7th graders at Sunview Middle School were helping to renovate a playground for the kindergartners at a nearby elementary school. City regulations require that the sand underneath the swings be at least 15 inches deep. The sand under both swing sets was only 12 inches deep when they started.

The rectangular area under the small swing set measures 9 feet by 12 feet and required 40 bags of sand to increase the depth by 3 inches. How many bags of sand will the students need to cover the rectangular area under the large swing set if it is 1.5 times as long and 1.5 times as wide as the area under the small swing set?

NGSSS vs. MAFS
Problems

8th Grade – NGSSS – MA.8.G.2.2

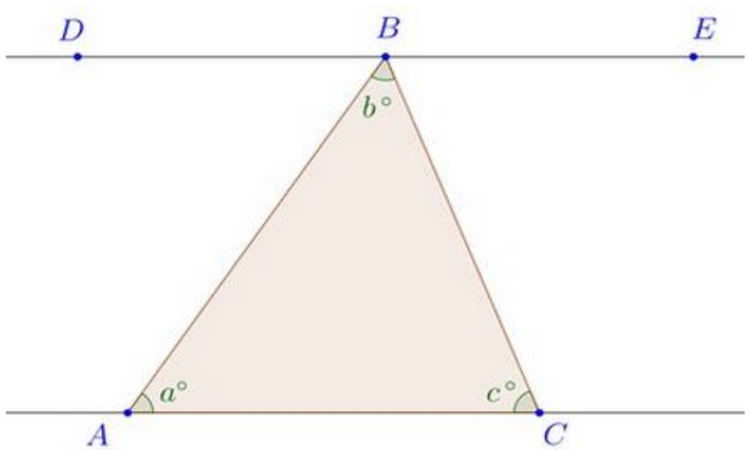
Anthony used strips of tape to guide him in painting an advertisement on a rectangular-shaped store window. The strips of tape are parallel and taped on the glass window. When he writes the message, he wants the letters to be slanted at 65° angles. The first word is shown below.



What is the measure, **in degrees**, of $\angle 1$?

8th Grade – MAFS.8.G.1.5

Given that $\overleftrightarrow{DE} \parallel \overleftrightarrow{AC}$ in the diagram below, prove that $a + b + c = 180$.



Explain why this result holds for any triangle, not just the one displayed above.

NGSSS vs. MAFS
Problems

Algebra NGSSS - MA.912.A.3.14

A corporation purchased a company for $50,000. The cost to run the company averages $1,000 per month and the revenue is $2,000 per month. The equations below model this situation, where ***t*** is the time, in months.

$$\textbf{Costs} = \$5{,}000 + 1{,}000t$$
$$\textbf{Revenue} = \$2{,}000\, t$$

How long will it take to break even (revenue = costs)?

A. 40 months
B. 45 months
C. 50 months
D. 55 months

Algebra – MAFS.912.A-REI.3.6

You are a representative for a cell phone company and it is your job to promote different cell phone plans.

a. Your boss asks you to visually display three plans and compare them so you can point out the advantages of each plan to your customers.

- Plan A costs a basic fee of $29.95 per month and 10 cents per text message
- Plan B costs a basic fee of $90.20 per month and has unlimited text messages
- Plan C costs a basic fee of $49.95 per month and 5 cents per text message
- All plans off unlimited calling
- Calling on nights and weekends are free
- Long distance calls are included

b. A customer wants to know how to decide which plan will save her the most money. Determine which plan has the lowest cost given the number of text messages a customer is likely to send.

Unwrapping a MAFS Content & Practice Standard

MA.5.NF.2.4. – Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problems. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

1. Sonia had $2\,{}^{1}\!/_{3}$ candy bars. She promised her brother that she would give him${}^{1}\!/_{2}$. How much will she have left after she gives her brother the amount promised?

2. If Mary ran 3 miles every four weeks she would reach her goal for the month. The first day of the first week she ran $1\,{}^{3}\!/_{4}$ miles. How many miles does she still need to run the first week?

3. Eli drank ${}^{3}\!/_{5}$ quart of milk and Javier drank ${}^{1}\!/_{10}$ less than Ellie. How much milk did they drink altogether?

4. Jerry was making two different types of cookies. One recipe needed ${}^{3}\!/_{4}$ cup of sugar and the other needed ${}^{2}\!/_{3}$ cup of sugar. How much sugar did he need to make both recipes?

5. Your teacher gave you ${}^{1}\!/_{7}$ of a bag of candy. She also gave your friend ${}^{1}\!/_{3}$ of the bag of candy. If you and your friend combined your candy, what fraction of the bag would you have? Estimate your answer and then calculate. How reasonable was your estimate?

Unwrapping a MAFS Content & Practice Standard

This document was created prior to the release of the MAFS item specifications. Upon its expected release in June 2014,

Domain: ______________________________ Cluster: ______________________________

Standard: MAFS.___.___.___.___ - ____

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This area may be modified upon the release of the MAFS item specifications in June 2014.

Cognitive Complexity Level: (circle one)	Level 3: Strategic Thinking & Complex Reasoning	Level 2: Basic Application of Skills & Concepts	Level 1: Recall	Item Type: (circle one)	Multiple-Choice	Multi-Select Items	Gridded-Response	Fill-In Response	Performance Tasks

Item Specification Sample Question

Complete the sample problem(s) in the space provided.

This area may be modified upon the release of the MAFS item specifications in June 2014.

Content Limits	Prior Knowledge/ Prerequisite Skills
Limits: What is the range of content knowledge and degree of difficulty that should be assessed in the items for the benchmark?	What knowledge, understanding, or reasoning is required to achieve this learning target?

This area may be modified upon the release of the MAFS item specifications in June 2014.

Practice Standards & Meaning Development	
What practice standard(s) will be utilized during this part of the lesson? What will students be doing in order to reach this practice standard(s)?	What teacher actions will contribute to the students' ability to engage in the practice standard(s)?
Manipulatives & Exploration	**Technology**
What manipulatives and exploration activities will provide support in teaching the conceptual understanding for this learning target? (*For example*: http://nlvm.usu.edu)	What technology resources will provide support to teach this learning target? Websites: ☐ illuminations. nctm.org ☐ explorelearning. com (Gizmos) Software: ☐ Geometer's Sketchpad ☐ Desmos Tools: ☐ Scientific/ Graphing Calc. ☐ Rulers, protractors, compasses, etc. Other: How will this technology be used to help build conceptual understanding?
Essential and Higher Order Questions	**Mastery Criteria**
Essential Question: • • Level 3 Complexity Questions (High): • • Level 2 Complexity Question (Mod.): • • Level 1 Complexity Question (Low): • •	What performance tasks or application products are required to demonstrate achievement on this learning target? (*For example*: https://www.illustrativemathematics.org/ and http://www.cpalms.org/) This area may be modified upon the release of the MAFS item specifications in June 2014.

MAFS: A Deep Dive into a Progression of the Standards

Standards

Directions: Cut out the following standards, standard codes, and sample problems to utilize with the mathematics progressions activity.

MATHEMATICS FLORIDA STANDARDS:

Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates. a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite. b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes. c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.
Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.
Interpret multiplication as scaling (resizing), by: a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication. b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.
Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*
Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.
Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*
Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram. a. Describe situations in which opposite quantities combine to make 0. For example, a hydrogen atom

has 0 charge because its two constituents are oppositely charged. b. Understand p + q as the number located a distance |q| from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts. c. Understand subtraction of rational numbers as adding the additive inverse, p – q = p + (–q). Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts. d. Apply properties of operations as strategies to add and subtract rational numbers.
Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.
Understand ordering and absolute value of rational numbers. a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. For example, interpret –3 > –7 as a statement that –3 is located to the right of –7 on a number line oriented from left to right. b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. For example, write –3 ºC > –7 ºC to express the fact that –3 ºC is warmer than –7 ºC. c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. For example, for an account balance of –30 dollars, write |–30| = 30 to describe the size of the debt in dollars. Distinguish comparisons of absolute value from statements about order. For example, recognize that an account balance less than –30 dollars represents a debt greater than 30 dollars.

MATHEMATICS FLORIDA STANDARD CODES:

MAFS.5.NF.2.3	MAFS.5.NF.2.5	MAFS.6.NS.3.5	MAFS.6.NS.3.6
MAFS.6.NS.3.7	MAFS.6.NS.3.8	MAFS.7.NS.1.1	MAFS.8.NS.1.2
MAFS.912.N-RN.2.3			

MAFS: A Deep Dive into a Progression of the Standards
Problems

"Fractions on the Number Line" Problem:

a. Find and label the numbers $\frac{4}{3}$, $\frac{5}{4}$, $-\frac{2}{3}$, and $-\frac{3}{4}$ on the number line.

b. For each of the following, state which inequality is true. Use the number line diagram to help explain your answers.

i. Is $\frac{4}{3} > \frac{5}{4}$, or is $\frac{4}{3} < \frac{5}{4}$?

ii. Is $-\frac{2}{3} > -\frac{3}{4}$, or is $-\frac{2}{3} < -\frac{3}{4}$?

c. Is $-\frac{3}{4}$ closer to 0 or is $\frac{5}{4}$? Explain how you know.

"Fractions on the Number Line" Solution 1:

Solution: Using a common denominator

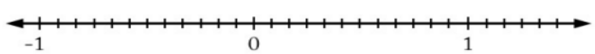
Another way to plot all of these numbers on the same number line is to first find a common denominator.

A common multiple of 3 and 4 is 12, so we can use 12 as a common denominator of $\frac{4}{3}$, $\frac{5}{4}$, $-\frac{2}{3}$ and $-\frac{3}{4}$ twelve.

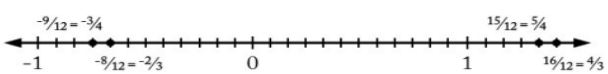
"Fractions on the Number Line" Solution 2:

$$\frac{4}{3} = \frac{16}{12} \qquad \frac{5}{4} = \frac{15}{12} \qquad -\frac{2}{3} = -\frac{8}{12} \qquad -\frac{3}{4} = -\frac{9}{12}$$

Now we can add more hash marks to our number line in increments of $\frac{1}{12}$.



Finally we can plot the numbers given.



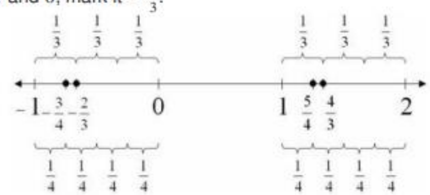
The remaining two parts of the question can be answered in a similar way to the first solution.

Solution: Reasoning about the position relative to 1 or -1

This solution depends on two insights:

- seeing the position of the fractions relative to 1 or -1
- seeing that thirds are bigger than fourths.

a. Mark the points -1,0,1,2 at regular intervals on a number line. For the positive fractions, notice that $\frac{5}{4}$ is $\frac{1}{4}$ more than $1 = \frac{4}{4}$ and that $\frac{4}{3}$ is $\frac{1}{3}$ more than $1 = \frac{3}{3}$. Choose a point one-fourth of the way between 1 and 2, label it $\frac{5}{4}$. To the right of that choose the point one-third of the way between 1 and 2 and mark the point $\frac{4}{3}$. Similarly, $-\frac{3}{4}$ is $\frac{1}{4}$ more than -1 and $-\frac{2}{3}$ is $\frac{1}{3}$ more than -1. Choose a point one-fourth of the way between -1 and 0, label it $-\frac{3}{4}$. A little to the right of it is the point $\frac{1}{3}$ of the way between -1 and 0; mark it $-\frac{2}{3}$.



b. Note: On a number line where positive numbers are to the right of zero and negative numbers are to the left of zero, numbers farther to the right are always greater than those to the left.

i. $\frac{4}{3} > \frac{5}{4}$

$\frac{4}{3}$ is to the right of $\frac{5}{4}$, so $\frac{4}{3}$ is greater than $\frac{5}{4}$.

ii. $-\frac{2}{3} > -\frac{3}{4}$

$-\frac{2}{3}$ is to the right of $-\frac{3}{4}$, so $-\frac{2}{3}$ is greater than $-\frac{3}{4}$.

c. $-\frac{3}{4}$ is closer to 0 than $\frac{5}{4}$. $-\frac{3}{4}$ is four units to the left of 0, while $\frac{5}{4}$ is five units to the right of 0.

“Irrational Numbers on the Number Line” Problem:

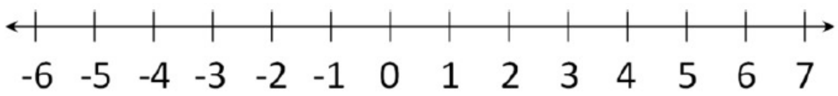
Without using your calculator, label approximate locations for the following numbers on the number line.

a. π

b. $-(\frac{1}{2} \times \pi)$

c. $2\sqrt{2}$

d. $\sqrt{17}$



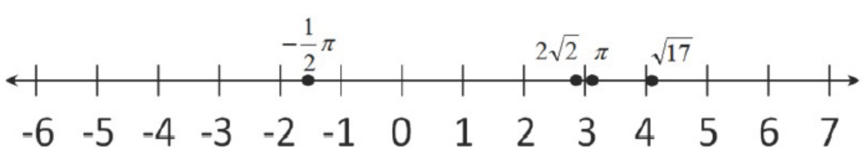
“Irrational Numbers on the Number Line” Solution:

a. π is slightly greater than 3.

b. $-(\frac{1}{2} \times \pi)$ is slightly less than -1.5.

c. $(2\sqrt{2})^2 = 4 \cdot 2 = 8$ and $3^2 = 9$, so $2\sqrt{2}$ is slightly less than 3.

d. $\sqrt{16} = 4$, so $\sqrt{17}$ is slightly greater than 4.



“Reasoning about Multiplication” Problem:

Your classmate Ellen says,

When you multiply by a number, you will always get a bigger answer. Look, I can show you.

Start with 9.	
Multiply by 5.	$9 \times 5 = 45$
The answer is 45, and	$45 > 9$
45 is bigger than 9.	

It even works for fractions.	
Start with $\frac{1}{2}$.	
Multiply by 4.	$\frac{1}{2} \times 4 = 2$
The answer is 2, and	$2 > \frac{1}{2}$
2 is bigger than $\frac{1}{2}$.	

Ellen's calculations are correct, but her rule does not always work.

For what numbers will Ellen's rule work? For what numbers will Ellen's rule not work? Explain and give examples.

“Reasoning about Multiplication” Solution:

Solution: Thinking of multiplication as scaling

Whenever you multiply a positive number by a factor greater than 1, the product will be larger than the original number. Both of Ellen's choices illustrate this principle.

Whenever you multiply a positive number by a positive factor less than 1, the product will be smaller than the original number. For example,

$$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}.$$

Both factors are less than 1, and the product is less than both factors.

Of course, whenever you multiply a number by 1, the product will be equal to the original number.

“Distances Between Houses” Problem:

Aakash, Bao Ying, Chris and Donna all live on the same street as their school, which runs from east to west.

- Aakash lives $5\frac{1}{2}$ blocks to the west.
- Bao Ying lives $4\frac{1}{4}$ blocks to the east.
- Chris lives $2\frac{3}{4}$ blocks to the west.
- Donna lives $6\frac{1}{2}$ blocks to the east.

a. Draw a picture that represents the positions of their houses along the street.

b. Find how far is each house from every other house?

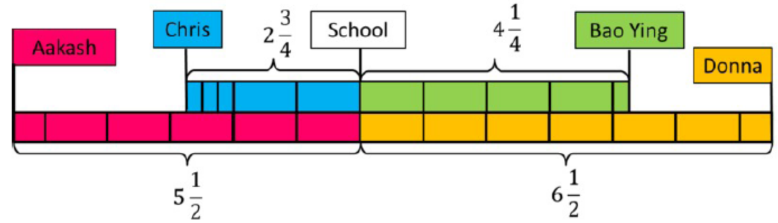
c. Represent the relative position of the houses on a number line, with the school at zero, points to the west represented by negative numbers, and points to the east represented by positive numbers.

d. How can you see the answers to part (b) on the number line? Using the numbers (some of which are positive and some negative) that label the positions of houses on the number line, represent these distances using sums or differences.

“Distances Between Houses” Solution:

Solution: 1

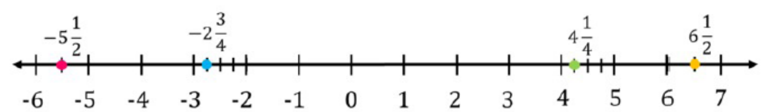
a. There are many ways to draw a picture that represents this situation. Here is one:



b. Here is a table that shows the distances between each of the student's houses.

	Bao Ying	Chris	Donna
Aakash	$9\frac{3}{4}$	$2\frac{3}{4}$	12
Bao Ying		7	$2\frac{1}{4}$
Chris			$7\frac{1}{4}$

c. The colors show which point corresponds to which person in the first picture:



d. The distance between the houses is represented by the distance between the points

“Operations with Rational and Irrational Numbers” Problem:

Experiment with sums and products of two numbers from the following list to answer the questions that follow:

$$5, \frac{1}{2}, 0, \sqrt{2}, -\sqrt{2}, \frac{1}{\sqrt{2}}, \pi.$$

Based on the above information, conjecture which of the statements is ALWAYS true, which is SOMETIMES true, and which is NEVER true?

a. The sum of a rational number and a rational number is rational.
b. The sum of a rational number and an irrational number is irrational.
c. The sum of an irrational number and an irrational number is irrational.
d. The product of a rational number and a rational number is rational.
e. The product of a rational number and an irrational number is irrational.
f. The product of an irrational number and an irrational number is irrational.

“Operations with Rational and Irrational Numbers” Solution:

Solution to part (c):

i. The sum of a rational number and a rational number is rational.

   Always true.

ii. The sum of a rational number and an irrational number is irrational.

   Always true.

iii. The sum of an irrational number and an irrational number is irrational.

   Only sometimes true (for instance, the sum of additive inverses like $\sqrt{2}$ and $-\sqrt{2}$ will be 0).

iv. The product of a rational number and a rational number is rational.

   Always true.

v. The product of a rational number and an irrational number is irrational.

   Not true -- but almost! This holds except when the rational number is zero.

vi. The product of an irrational number and an irrational number is irrational.

   Only sometimes true (for instance, the product of multiplicative inverses like $\sqrt{2}$ and $\frac{1}{\sqrt{2}}$ will be 1).

“How Much Pie?” Problem:

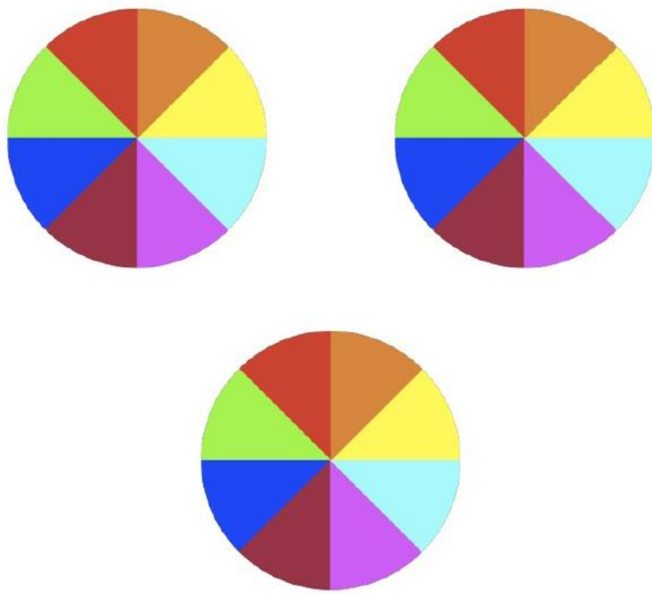
After a class potluck, Emily has three equally sized apple pies left and she wants to divide them into eight equal portions to give to eight students who want to take some pie home.

a. Draw a picture showing how Emily might divide the pies into eight equal portions. Explain how your picture shows eight equal portions.

b. What fraction of a pie will each of the eight students get?

c. Explain how the answer to (b) is related the division problem $3 \div 8$.

“How Much Pie?” Solution:



Because these pies are all the same size and they are all apple pies, Emily does not need to give each student one piece of each of the three pies: two or three pieces of the same pie could go to one student. This picture, however, shows clearly that the pies have been divided into eight equal portions. If multiple pieces of a particular pie were to go to the same student, it would be necessary to analyze the picture more closely and count how many slices of pie each student received to check that it has been divided evenly.

c. If 3 pies are divided into 8 equal portions, then 8 of these portions makes 3 pies, a fact that is clearly illustrated in part (a). We can write this in symbols if we use a question mark to represent the amount of pie in one portion:

$$8 \times ? = 3$$

When we know a factor and the product, we can find the other factor by dividing:

$$3 \div 8 = ?$$

So one person's portion is whatever we get when we divide 3 by 8. In part (b), we saw that one portion is $\frac{3}{8}$. So that means that

$$3 \div 8 = \frac{3}{8}.$$

<u>“Mile High” Problem:</u>

Denver, Colorado is called “The Mile High City” because its elevation is 5280 feet above sea level. Someone tells you that the elevation of Death Valley, California is -282 feet.

a. Is Death Valley located above or below sea level? Explain.

b. How many feet higher is Denver than Death Valley?

c. What would your elevation be if you were standing near the ocean?

<u>“Mile High” Solution:</u>

a. Death Valley is located below sea level. We know this because its elevation is negative. Sea level is the base for measuring elevation. Sea level elevation is defined as 0 ft. All other elevations are measured from sea level. Those places on Earth that are above sea level have positive elevations, and those places on Earth that are below sea level have negative elevations. Thus, Death Valley, with an elevation of -282 feet, is located below sea level.

b. To find out how much higher Denver is than Death Valley, we can reason as follows:

Death Valley is 282 feet below sea level. Denver is 5280 above sea level. So to go from Death Valley to Denver, you would go up 282 feet to get to sea level and then go up another 5280 feet to get to Denver for a total of

$$282 + 5280 = 5562.$$

feet. Thus, Denver, Colorado is 5562 feet higher than Death Valley, California.

c. If you were standing near the ocean, your elevation would be close to zero. Depending on how high or low the tide is and where exactly you are standing, your elevation could be as low as -50 feet (or as high as 50 feet) if you are at the edge of a very low tide (or a very high tide, respectively) at the Bay of Fundy.

<u>“Comparing Temperatures” Problem:</u>

a. Here are the low temperatures (in Celsius) for one week in Juneau, Alaska:

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
5	-1	-6	-2	3	7	0

Arrange them in order from coldest to warmest temperature.

b. On a winter day, the low temperature in Anchorage was 23 degrees below zero (in $^{\circ}C$) and the low temperature in Minneapolis was 14 degrees below zero (in $^{\circ}C$). Sophia wrote,

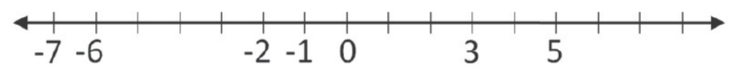
Minneapolis was colder because $-14 < -23$.

Is Sophia correct? Explain your answer.

c. The lowest temperature ever recorded on earth was $-89^{\circ}C$ in Antarctica. The average temperature on Mars is about $-55^{\circ}C$. Which is warmer, the coldest temperature on earth or the average temperature on Mars? Write an inequality to support your answer.

<u>“Comparing Temperatures” Solution:</u>

a. Let’s begin by plotting them all on the same number line.



The number line has positive numbers to the right of zero and negative numbers to the left of zero. This means that numbers father to the right are always greater than those to the left. In terms of temperature, the coldest temperature (the least number) is all the way to the left, and the warmest temperature (the greatest number) is all the way to the right.

We can now list the temperatures from coldest to warmest:

$$-7, -6, -2, -1, 0, 3, 5$$

b. Sophia is incorrect. It is common for students to compare negative numbers as if they were positive and to assume that the one with the greatest magnitude is the greatest number. However, -23 is to the left of -14 on the number line, and so it is less than -14. Thus

$$-23 < -14$$

and Anchorage was colder.

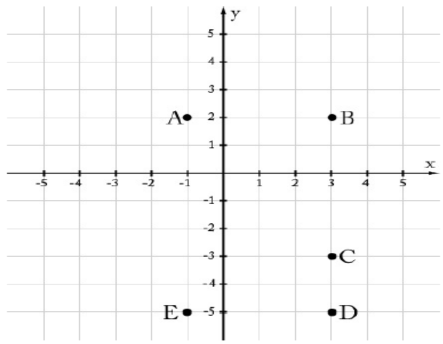
c. Again, the coldest temperature corresponds to the least number. So the warmest temperature corresponds to the greatest number. Since

$$-55 > -89$$

the average temperature on Mars is warmer than the coldest temperature on Earth.

“Distances between Points” Problem:

Some points are shown in the coordinate plane below.



a. What is the distance between points B & C?

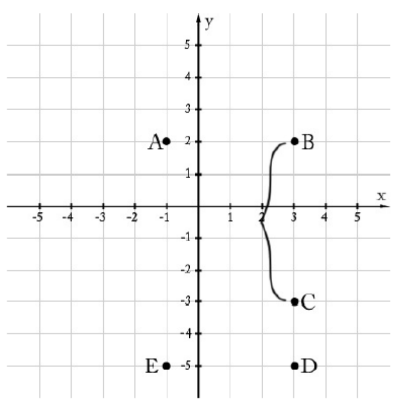
b. What is the distance between points D & B?

c. What is the distance between points D & E?

d. Which of the points shown above are 4 units away from $(-1,-3)$ and 2 units away from $(3,-1)$?

“Distances between Points” Solution:

a. The distance between points B and C is 5 units, as indicated below.



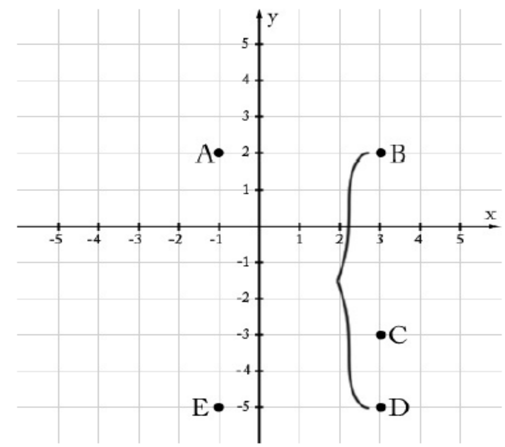
We can determine the distance between points B and C by counting the units between the two points since they are on the same vertical line which happens to be a grid line.

We can also note that the coordinates for B are $(3,2)$ and so it is |2| units above the x-axis. Similarly, the coordinates for C are $(3,-3)$, so it is |-3| units below the x-axis. Thus, the distance between B and C is

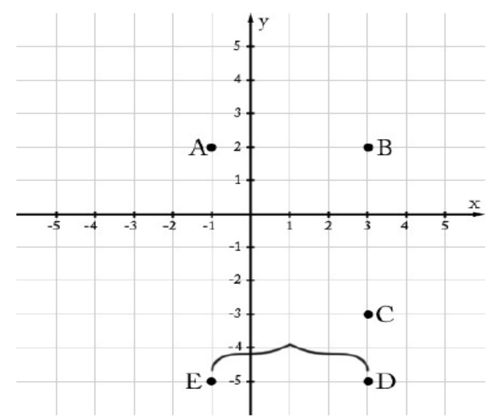
$$|2|+|-3|=5$$

units.

b. The distance between points D and B is 7 units, as indicated below.

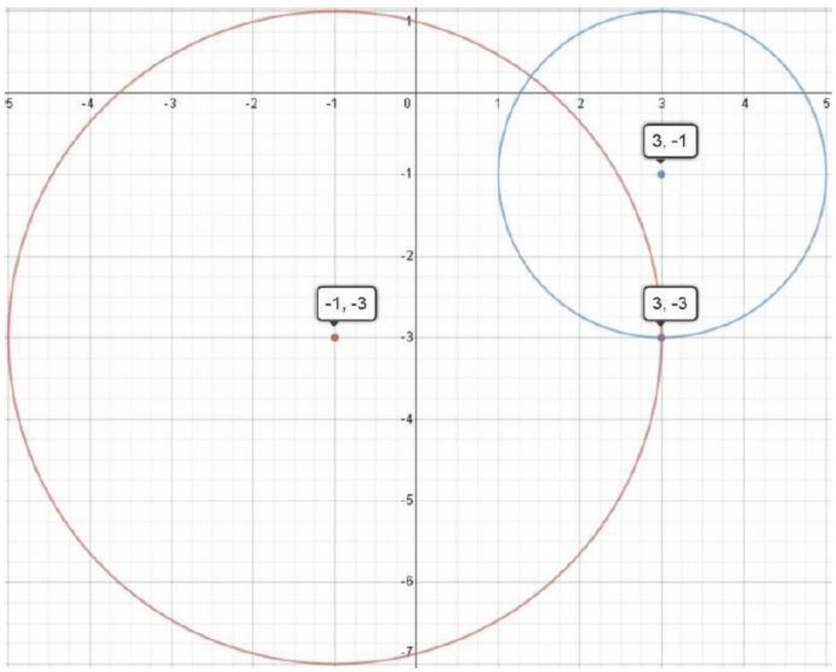


c. The distance between points D and E is 4 units, as indicated below.



d. Point C is the only one of points $A, \dots E$ that is both 4 units away from $(-1,-3)$ and 2 units away from $(3,-1)$. There is another point that satisfies these two criteria (which is the other intersection point of the circle of radius 4 and circle of radius 2 shown

below), but it does not have integer coordinates and is not one of the points identified in the task statement.



Progressions for the Common Core State Standards in Mathematics (draft)

4 July 2013

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For updates and more information about the Progressions, see http://ime.math.arizona.edu/progressions.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

The Number System, 6–8

Overview

In Grades 6–8, students build on two important conceptions which have developed throughout K–5, in order to understand the rational numbers as a number system. The first is the representation of whole numbers and fractions as points on the number line, and the second is a firm understanding of the properties of operations on whole numbers and fractions.

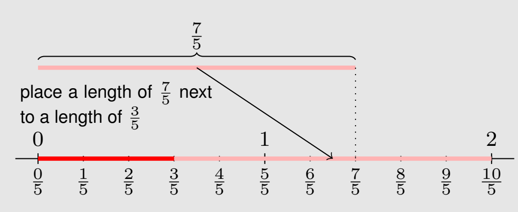
Representing numbers on the number line In early grades, students see whole numbers as counting numbers, Later, they also understand whole numbers as corresponding to points on the number line. Just as the 6 on a ruler measures 6 inches from the 0 mark, so the number 6 on the number line measures 6 units from the origin. Interpreting numbers as points on the number line brings fractions into the family as well; fractions are seen as measurements with new units, creating by partitioning the whole number unit into equal pieces. Just as on a ruler we might measure in tenths of an inch, on the number line we have halves, thirds, fifths, sevenths; the number line is a sort of ruler with every denominator. The denominators 10, 100, etc. play a special role, partioning the number line into tenths, hundredths, etc., just as a metric ruler is partioned into centimeters and millimeters.

Starting in Grade 2 students see addition as concatenation of lengths on the number line.[2.MD.6] By Grade 4 they are using the same model to represent the sum of fractions with the same denominator: $\frac{3}{5}+\frac{7}{5}$ is seen as putting together a length that is 3 units of one fifth long with a length that is 7 units of one fifth long, making 10 units of one fifths in all. Since there are five fifths in 1 (that's what it means to be a fifth), and 10 is 2 fives, we get $\frac{3}{5}+\frac{7}{5}=2$. Two fractions with different denominators are added by representing them in terms of a common unit.

Representing sums as concatenated lengths on the number line is important because it gives students a way to think about addition that makes sense independently of how numbers are represented symbolically. Although addition calculations may look different for numbers represented in base ten and as fractions, addition is the

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

Representing $\frac{3}{5}$ and $\frac{7}{5}$ on the number line



same operation in each case. Furthermore, the concatenation model of addition extends naturally to negative numbers in Grade 7.

Properties of operations The number line provides a representation that can be used to building understanding of sums and differences of rational numbers. However, building understanding of multiplication and division of rational numbers relies on a firm understanding of properties of operations. Although students have not necessarily been taught formal names for these properties, they have used them repeatedly in elementary school and have been with reasoning with them. The commutative and associative properties of addition and mutiplication have, in particular, been their constant friends in working with strategies for addition and multiplication.[1.OA.3, 3.OA.5]

The existence of the multiplicative identity (1) and multiplicative inverses start to play important roles as students learn about fractions. They might see fraction equivalence as confirming the identity rule for fractions. In Grade 4 they learn about fraction equivalence

$$\frac{n \times a}{n \times b} = \frac{a}{b}$$

and in Grade 5 they relate this to multiplication by 1

$$\frac{n \times a}{n \times b} = \frac{n}{n} \times \frac{a}{b} = 1 \times \frac{a}{b},$$

thus confirming that the identity rule

$$1 \times \frac{a}{b} = \frac{a}{b}$$

works for fractions.[5.NF.5]

As another example, the commutative property for multiplication plays an important role in understanding multiplication with fractions. For example, although

$$5 \times \frac{1}{2} = \frac{5}{2}$$

can be made sense of using previous understandings of whole number multiplication as repeated addition, the other way around,

$$\frac{1}{2} \times 5 = \frac{5}{2},$$

seems to come from a different source, from the meaning of phrases such as "half of" and a mysterious acceptance that "of" must mean multiplication. A more reasoned approach would be to observe that if we want the commutative property to continue to hold, then we must have

$$\frac{1}{2} \times 5 = 5 \times \frac{1}{2} = \frac{5}{2},$$

Properties of Operations on Rational Numbers

Properties of Addition

1. Commutative Property. For any two rational numbers a and b, $a + b = b + a$.
2. Associative Property. For any three rational numbers a, b and c, $(a + b) + c = a + (b + c)$.
3. Existence of Identity. The number 0 satisfies $0 + a = a = a + 0$.
4. Existence of Additive Inverse. For any rational number a, there is a number $-a$ such that $a + (-a) = 0$.

Properties of Multiplication

1. Commutative Property. For any two rational numbers a and b, $a \times b = b \times a$.
2. Associative Property. For any three rational numbers a, b and c, $(a \times b) \times c = a \times (b \times c)$.
3. Existence of Identity. The number 1 satisfies $1 \times a = a = a \times 1$.
4. Existence of Multiplicative Inverse. For every non-zero rational number a, there is a rational number $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$.

The Distributive Property

For rational numbers a, b and c, one has $a \times (b + c) = a \times b + a \times c$.

1.OA.3 Apply properties of operations as strategies to add and subtract.[1]

3.OA.5 Apply properties of operations as strategies to multiply and divide.[2]

5.NF.5 Interpret multiplication as scaling (resizing), by:

a ...

b ...and relating the principle of fraction equivalence $\frac{a}{b} = \frac{n \times a}{n \times b}$ to the effect of multiplying $\frac{a}{b}$ by 1.

and that $\frac{5}{2}$ is indeed "half of five," as we have understood in Grade 5.[5.NF.3]

When students extend their conception of multiplication to include negative rational numbers, the properties of operations become crucial. The rule that the product of negative numbers is positive, often seen as mysterious, is the result of extending the properties of operations (particularly the distributive property) to rational numbers.

[5.NF.3] Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

Grade 6

As Grade 6 dawns, students have a firm understanding of place value and the properties of operations. On this foundation they are ready to start using the properties of operations as tools of exploration, deploying them confidently to build new understandings of operations with fractions and negative numbers. They are also ready to complete their growing fluency with algorithms for the four operations.

Apply and extend previous understandings of multiplication and division to divide fractions by fractions In Grade 6 students conclude the work with operations on fractions, started in Grade 4, by computing quotients of fractions.[6.NS.1] In Grade 5 students divided unit fractions by whole numbers and whole numbers by unit fractions, two special cases of fraction division that are relatively easy to conceptualize and visualize.[5.NF.7ab] Dividing a whole number by a unit fraction can be conceptualized in terms of the measurement interpretation of division, which conceptualizes $a \div b$ as the the measure of of a by units of length b on the number line, that is, the solution to the multiplication equation $a = ? \times b$. Dividing a unit fraction by a whole number can be interpreted in terms of the sharing interpretation of division, which conceptualizes $a \div b$ as the size of a share when a is divided into b equal shares, that is, the solution to the multiplication equation $a = b \times ?$.

Now in Grade 6 students develop a general understanding of fraction division. They can use story contexts and visual models to develop this understanding, but also begin to move towards using the relation between division and multiplication.

For example, they might use the measurement interpretation of division to see that $\frac{8}{3} \div \frac{2}{3} = 4$, because 4 is 4 is how many $\frac{2}{3}$ there are in $\frac{8}{3}$. At the same time they can see that this latter statement also says that $4 \times \frac{2}{3} = \frac{8}{3}$. This multiplication equation can be used to obtain the division equation directly, using the relation between multiplication and division.

Quotients of fractions that are whole number answers are particularly suited to the measurement interpretation of division. When this interpretation is used for quotients of fractions that are not whole numbers, it can be rephrased from "how many times does this go into that?" to "how much of this is in that?" For example,

$$\frac{2}{3} \div \frac{3}{4}$$

can be interpreted as how many $\frac{3}{4}$-cup servings are in $\frac{2}{3}$ of a cup of yogurt, or as how much of a $\frac{3}{4}$-cup serving is in $\frac{2}{3}$ of a cup of yogurt. Although linguistically different the two questions are mathematically the same. Both can be visualized as in the margin and expressed using a multiplication equation with an unknown for the

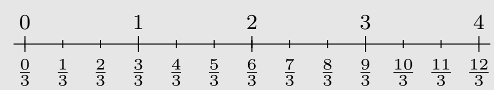
6.NS.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.

5.NF.7 Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

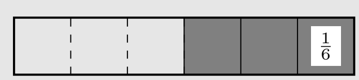
a Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.

b Interpret division of a whole number by a unit fraction, and compute such quotients.

Visual models for division of whole numbers by unit fractions and unit fractions by whole numbers

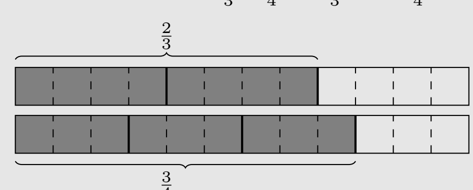


Reasoning on a number line using the measurement interpretation of division: there are 3 parts of length $\frac{1}{3}$ in the unit interval, therefore there are 4×3 parts of length $\frac{1}{3}$ in the interval from 0 to 4, so the number of times $\frac{1}{3}$ goes into 4 is 12, that is $4 \div \frac{1}{3} = 4 \times 3 = 12$.



Reasoning with a fraction strip using the sharing interpretation of division: the strip is the whole and the shaded area is $\frac{1}{2}$ of the whole. If the shaded area is divided into 3 equal parts, then 2×3 of those parts make up the whole, so $\frac{1}{2} \div 3 = \frac{1}{2 \times 3} = \frac{1}{6}$.

Visual model for $\frac{2}{3} \div \frac{3}{4}$ and $\frac{2}{3} = ? \times \frac{3}{4}$



We find a common unit for comparing $\frac{2}{3}$ and $\frac{3}{4}$ by dividing each $\frac{1}{3}$ into 4 parts and each $\frac{1}{4}$ into 3 parts. Then $\frac{2}{3}$ is 8 parts when $\frac{3}{4}$ is divided into 9 equal parts, so $\frac{2}{3} = \frac{8}{9} \times \frac{3}{4}$, which is the same as saying that $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$.

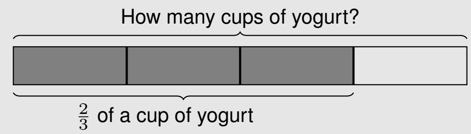
first factor:

$$\frac{2}{3} = ? \times \frac{3}{4}.$$

The same division problem can be interpreted using the sharing interpretation of division: how many cups are in a full container of yogurt when $\frac{2}{3}$ of a cup fills $\frac{3}{4}$ of the container. In other words, $\frac{3}{4}$ of what amount is equal to $\frac{2}{3}$ cups? In this case, $\frac{2}{3} \div \frac{3}{4}$ is seen as the solution to a multiplication equation with an unknown as the second factor:

$$\frac{3}{4} \times ? = \frac{2}{3}.$$

Visual model for $\frac{2}{3} \div \frac{3}{4}$ and $\frac{3}{4} \times ? = \frac{2}{3}$



The shaded area is $\frac{3}{4}$ of the entire strip. So it is 3 parts of a division of the strip into 4 equal parts. Another way of seeing this is that the strip is 4 parts of a division of the shaded area into 3 equal parts. That is, the strip is $\frac{4}{3}$ times the shaded part. So $? = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}$.

There is a close connection between the reasoning shown in the margin and reasoning about ratios; if two quantities are in the ratio $3 : 4$, then there is a common unit so that the first quantity is 3 units and the second quantity is 4 units. The corresponding unit rate is $\frac{3}{4}$, and the first quantity is $\frac{3}{4}$ times the second quantity. Viewing the situation the other way around, with the roles of the two quantities interchanged, the same reasoning shows that the second quantity is $\frac{4}{3}$ times the first quantity. Notice that this leads us directly to the invert-and-multiply for fraction division: we have just reasoned that the ? in the equation above must be equal to $\frac{4}{3} \times \frac{2}{3}$, which is exactly what the rules gives us for $\frac{2}{3} \div \frac{4}{3}$.[6.NS.1]

6.NS.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.

The invert-and-multiply rule can also be understood algebraically as a consequence of the general rule for multiplication of fractions. If $\frac{a}{b} \div \frac{c}{d}$ is is defined to be the missing factor in the multiplication equation

$$? \times \frac{c}{d} = \frac{a}{b}$$

then the fraction that does the job is

$$? = \frac{ad}{bc},$$

as we can verify by putting it into the multiplication equation and using the already known rules of fraction multiplication and the properties of operations:

$$\frac{ad}{bc} \times \frac{c}{d} = \frac{(ad)c}{(bc)d} = \frac{a(cd)}{b(cd)} = \frac{a}{b} \times \frac{cd}{cd} = \frac{a}{b}.$$

Compute fluently with multi-digit numbers and find common factors and multiples In Grade 6 students consolidate the work of earlier grades on operations with whole numbers and decimals by becoming fluent in the four operations on these numbers.[6.NS.2, 6.NS.3] Much of the foundation for this fluency has been laid in earlier grades. They have known since Grade 3 that whole numbers are fractions[3.NF.3c] and since Grade 4 that decimal notation is a way of writing fractions with denominator equal to a power of 10;[4.NF.6] by Grade 6 they start to see whole numbers, decimals and fractions

6.NS.2 Fluently divide multi-digit numbers using the standard algorithm.

6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

3.NF.3c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

4.NF.6 Use decimal notation for fractions with denominators 10 or 100.

not as wholly different types of numbers but as as part of the same number system, represented by the number line.

In many traditional treatments of fractions greatest common factors occur in reducing a fraction to lowest terms, and least common multiples occur in adding fractions. As explained in the Fractions Progression, neither of these activities is treated as a separate topic in the standards. Indeed, insisting that finding a least common multiple is an essential part of adding fractions can get in the way of understanding the operation, and the excursion into prime factorization and factor trees that is often entailed in these topics can be time-consuming and distract from the focus of K–5. In Grade 6, however, students experience a modest introduction to the concepts[6.NS.4] and put the idea of greatest common factor to use in a rehearsal for algebra, where they will need to see, for example, that $3x^2 + 6x = 3x(x + 2)$.

6.NS.4 Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor.

Apply and extend previous understandings of numbers to the system of rational numbers In Grade 6 the number line is extended to include negative numbers. Students initially encounter negative numbers in contexts where it is natural to describe both the magnitude of the quantity, e.g. vertical distance from sea level in meters, and the direction of the quantity (above or below sea level).[6.NS.5] In some cases 0 has an essential meaning, for example that you are at sea level; in other cases the choice of 0 is merely a convention, for example the temperature designated as 0° in Farenheit or Celsius. Although negative integers might be commonly used as initial examples of negative numbers, the Standards do not introduce the integers separately from the entire system of rational numbers, and examples of negative fractions or decimals can be included from the beginning.

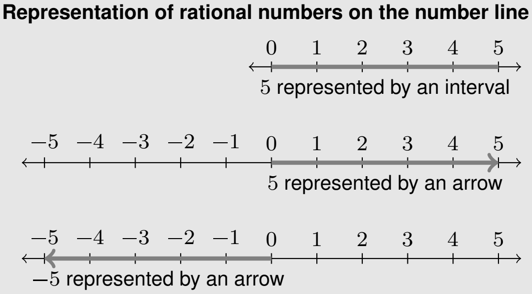
6.NS.5 Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

6.NS.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

a Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.

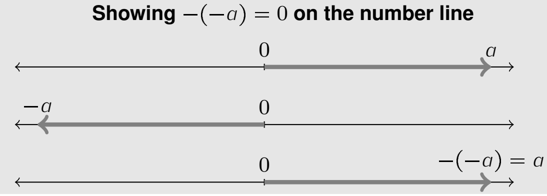
Directed measurement scales for temperature and elevation provide a basis for understanding positive and negative numbers as having a positive or negative direction on the number line.[6.NS.6a] Previous understanding of numbers on the number line related the position of the number to measurement: the number 5 is located at the endpoint of an line segment 5 units long whose other endpoint is at 0. Now the line segments acquire direction; starting at 0 they can go in either the positive or the negative direction. These directed numbers can be represented by putting arrows at the endpoints of the line segments.

Representation of rational numbers on the number line



Students come to see $-p$ as the opposite of p, located an equal distance from 0 in the opposite direction. In order to avoid the common misconception later in algebra that any symbol with a negative sign in front of it should be a negative number, it is useful for students to see examples where $-p$ is a positive number, for example if $p = -3$ then $-p = -(-3) = 3$. Students come to see the operation of putting a negative sign in front of a number as flipping the direction of the number from positive to negative or negative to

Showing $-(-a) = 0$ on the number line



positive. Students generalize this understanding of the meaning of the negative sign to the coordinate plane, and can use it in locating numbers on the number line and ordered pairs in the coordinate plane.[6.NS.6bc]

With the introduction of negative numbers, students gain a new sense of ordering on the number line. Whereas statements like $5 < 7$ could be understood in terms of having more of or less of a certain quantity—"I have 5 apples and you have 7, so I have fewer than you"—comparing negative numbers requires closer attention to the relative positions of the numbers on the number line rather than their magnitudes.[6.MS.7a] Comparisons such as $-7 < -5$ can initially be confusing to students, because -7 is further away from 0 than -5, and is therefore larger in magnitude. Referring back to contexts in which negative numbers were introduced can be helpful: 7 meters below sea level is lower than 5 meters below sea level, and $-7°$ F is colder than $-5°$ F. Students are used to thinking of colder temperatures as lower than hotter temperatures, and so the mathematically correct statement also makes sense in terms of the context.[6.NS.7b]

At the same time, the prior notion of distance from 0 as a measure of size is still present in the notion of absolute value. To avoid confusion it can help to present students with contexts where it makes sense both to compare the order of two rational numbers and to compare their absolute value, and where these two comparisons run in different directions. For example, someone with a balance of \$100 in their bank account has more money than someone with a balance of $-\$1000$, because $100 > -1000$. But the second person's debt is much larger than the first person's credit $|-1000| > |100|$.[6.NS.7cd]

This understanding is reinforced by extension to the coordinate plane.[6.NS.8]

b Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

c Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

6.NS.7 Understand ordering and absolute value of rational numbers.

a Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram.

b Write, interpret, and explain statements of order for rational numbers in real-world contexts.

c Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation.

d Distinguish comparisons of absolute value from statements about order.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Grade 7

Addition and subtraction of rational numbers In Grade 6 students learned to locate rational numbers on the number line; in Grade 7 they extend their understanding of operations with fractions to operations with rational numbers. Whereas previously addition was represented by concatenating the line segments together, now the the line segments have directions, and therefore a beginning and an end. When concatenating these directed line segments, we start the second line segment at the end of the first one. If the second line segment is going in the opposite direction to the first, it can backtrack over the first, effectively cancelling part or all of it out.[7.NS.1b] Later in high school, if students encounter vectors, they will be able to see this as one-dimensional vector addition.

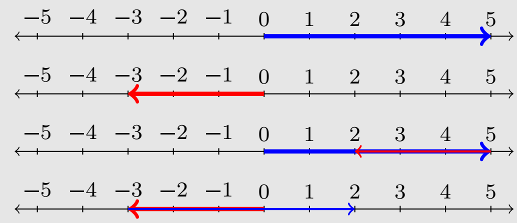
A fundamental fact about addition of rational numbers is that $p + (-p) = 0$ for any rational number p; in fact, this is a new property of operations that comes into play when negative numbers are introduced. This property can be introduced using situations in which the equation makes sense in a context.[7.NS.1a] For example, the operation of adding an integer could be modeled by an elevator moving up or down a certain number of floors. It can also be shown using the directed line segment model of addition on the number, as shown in the margin.[7.NS.1b]

It is common to use colored chips to represent integers, with one color representing positive integers and another representing negative integers, subject to the rule that chips of different colors cancel each other out; thus, a number is not changed if you take away or add such a pair. This is rather a different representation than the number line. On the number line, the equation $p+(-p) = 0$ follows from the definition of addition using directed line segments. With integer chips, the equation $p + (-p) = 0$ is true by definition since it it is encoded in the rules for manipulating the chips. Also implicit in the use of chips is that the commutative and associative properties extend to addition of integers, since combining chips can be done in any order.

However, the integer chips are not suited to representing rational numbers that are not integers. Whether such chips are used or not, the Standards require that students eventually understand location and addition of rational numbers on the number line. With the number line model, showing that the properties of operations extend to rational numbers requires some reasoning. Although it is not appropriate in Grade 6 to insist that all the properties be proved proved to hold in the number line or chips model, experimenting with them in these models is a good venue for reasoning (MP.2).[7.NS.1d]

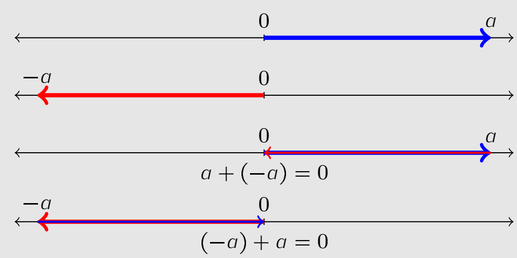
Subtraction of rational numbers is defined the same way as for positive rational numbers: $p - q$ is defined to be the missing addend in $q + ? = p$. For example, in earlier grades, students understand $5 - 3$ as the missing addend in $3 + ? = 5$. On the number line, it

Showing $5 + (-3) = 2$ and $-3 + 5 = 2$ on the number line



The number 5 is represented by the blue arrow pointing right from 0, and the number −3 is represented by the red arrow pointing left from 0. To add $5 + (-3)$ we place the arrow for 5 down first then attach the arrow for −3 to its endpoint. To add $-3 + 5$ we place the arrow for −3 down first then attach the arrow for 5 to its endpoint.

Showing $a + (-a) = 0$, and $(-a) + a = 0$ on the number line



7.NS.1 Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

a Describe situations in which opposite quantities combine to make 0.

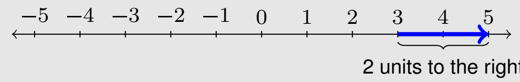
b Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.

d Apply properties of operations as strategies to add and subtract rational numbers.

is represented as the distance from 3 to 5. Or, with our newfound emphasis on direction on the number line, we might say that it is how you get from 3 from 5; by going two units to the right (that is, by adding 2).

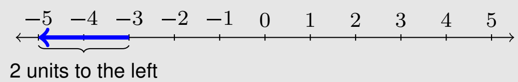
In Grade 6 students apply the same understanding to $(-5) - (-3)$. It is the missing addend in $(-3) + ? = -5$, or how you get from -3 to -5. Since -5 is two units to the left of -3 on the number line, the missing addend is -2.

Showing $5 - 3 = 2$ on the number line.



You get from 3 to 5 by adding 2, so $5 - 3 = 2$.

Showing $(-5) - (-3) = -2$ on the number line.



You get from -3 to -5 by adding -2, so $(-5) - (-3) = -2$

With the introduction of direction on the number line, there is a distinction between the distance from a and b and how you get from a to b. The distance from -3 to -5 is 2 units, but the instructions how to get from -3 to -5 are "go two units to the left." The distance is a positive number, 2, wherease "how to get there" is a negative number -2. In Grade 6 we introduce the idea of absolute value to talk about the size of a number, regardless of its sign. It is always a positive number or zero. If p is positive, then its absolute value $|p|$ is just p; if p is negative then $|p| = -p$. With this interpretation we can say that the absolute value of $p - q$ is just the distance from p to q, regardless of direction.[7.NS.1c]

7.NS.1c Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

c Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

Understanding $p - q$ as a missing addend also helps us see that $p + (-q) = p - q$.[7.NS.1c] Indeed, $p - q$ is the missing number in

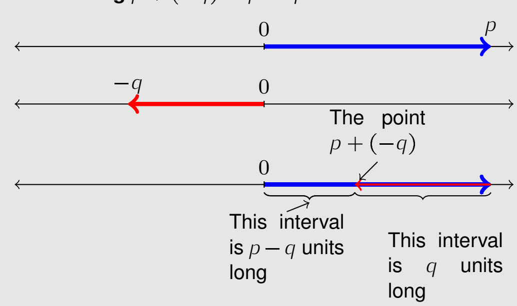
$$q + ? = p$$

and $p + (-q)$ satisfies the description of being that missing, number:

$$q + (p + (-q)) = p + (q + (-q)) = p + 0 = p.$$

The figure in the margin illustrates this in the case where p and q are positive and $p > q$.

Showing $p + (-q) = p - q$ on the number line



The red directed interval representing $-q$ is q units long, so the remaining part of the blue directed interval representing p is $p - q$ units long.

Multiplication and division of rational numbers Hitherto we have been able to come up with visual models to represent rational numbers, and the operations of addition and subtraction on them. This starts to break down with multiplication and division, and students must rely increasingly on the properties of operations to build the necessary bridges from their previous understandings to situations where one or more of the numbers might be negative.

For example, multiplication of a negative number by a positive whole number can still be understood as before; just as 5×2 can be understood as $2 + 2 + 2 + 2 + 2 = 10$, so 5×-2 can be understood as $(-2) + (-2) + (-2) + (-2) + (-2) = -10$. We think of 5×2 as five jumps to the right on the number line, starting at 0, and we think of $5 \times (-2)$ as five jumps to the left.

But what about $\frac{3}{4} \times -2$, or -5×-2? Perhaps the former can be undersood as going $\frac{3}{4}$ of the way from 0 to -2, that is $-\frac{3}{2}$. For the latter, teachers sometimes come up with metaphors involving going backwards in time or repaying debts. But in the end these metaphors do not explain why $-5 \times -2 = 10$. In fact, this is a choice we make, not something we can justify by appeals to real world situations.

Why do we make the choice of saying that a negative times a negative is positive? Because we want to extend the operation of multiplication to rational number is such a way that *all* of the properties of operations are satisfied.[7.NS.2a] In particular, the property that really makes a difference here is the distributive property. If you want to be able to say that

$$4 \times (5 + (-2)) = 4 \times 5 + 4 \times (-2),$$

you have to say that $4 \times (-2) = -8$, because the number on the left is 12 and the first addend on the right is 20. This leads to the rules

$$\text{positive}\times\text{negative} = \text{negative} \quad \text{and} \quad \text{negative}\times\text{positive} = \text{negative}.$$

If you want to be able to say that

$$(-4) \times (5 + (-2)) = (-4) \times 5 + (-4) \times (-2),$$

then you have to say that $(-4) \times (-2) = 8$, since now we know that the number on the left is -12 and the first addend on the right is -20. This leads to the rule

$$\text{negative} \times \text{negative} = \text{positive}.$$

Why is it important to maintain the distributive property? Because when students get to algebra, they use it all the time. They must be able to say $-3x - 6y = -3(x + 2y)$ without worrying about whether x and y are positive or negative.

The rules about moving negative signs around in a product result from the rules about multiplying negative and positive numbers. Think about the various possibilities for p and q in

$$p \times (-q) = (-p) \times q = -pq.$$

If p and q are both positive, then this just a restatement of the rules above. But it still works if, for example, p is negative and q is positive. In that case it says

$$\text{negative} \times \text{negative} = \text{positive} \times \text{positive} = \text{positive}.$$

Just as the relationship between addition and subtraction helps students understand subtraction of rational numbers, so the relationship between multiplication and division helps them understand division. To calculate $-8 \div 4$, students recall that $(-2) \times 4 = -8$, and so $-8 \div 4 = -2$. By the same reasoning,

$$-8 \div 5 = -\frac{8}{5} \quad \text{because} \quad -\frac{8}{5} \times 5 = -8.$$

7.NS.2a Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

a Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

This means it makes sense to write

$$-8 \div 5 \quad \text{as} \quad \frac{-8}{5}.$$

Until this point students have not seen fractions where the numerator or denominator could be a negative integer. But working with the corresponding multiplication equations allows students to make sense of such fractions. In general, they see that[7.NS.2b]

$$-\frac{p}{q} = \frac{-p}{q} = \frac{p}{-q}$$

for any integers p and q, with $q \neq 0$.

Again, using multiplication as a guide, students can extend division to rational numbers that are not integers.[7.NS.2c] For example

$$\frac{2}{3} \div (-\frac{1}{2}) = -\frac{4}{3} \quad \text{because} \quad -\frac{4}{3} \times -\frac{1}{2} = \frac{2}{3}.$$

And again it makes sense to write this division as a fraction:

$$\frac{\frac{2}{3}}{-\frac{1}{2}} = -\frac{4}{3} \quad \text{because} \quad -\frac{4}{3} \times -\frac{1}{2} = \frac{2}{3}.$$

Note that this is an extension of the fraction notation to a situation it was not originally designed for. There is no sense in which we can think of

$$\frac{\frac{2}{3}}{-\frac{1}{2}}$$

as $\frac{2}{3}$ parts where one part is obtained by dividing the line segment from 0 to 1 into $-\frac{1}{2}$ equal parts! But the fact that the properties of operations extend to rational numbers means that calculations with fractions extend to these so-called complex fractions $\frac{p}{q}$, where p and q could be rational numbers, not only integers. By the end of Grade 7, students are solving problems involving complex fractions.[7.NS.3]

Decimals are special fractions, those with denominator 10, 100, 1000, etc. But they can also be seen as a special sort of measurement on the number line, namely one that you get by partitioning into 10 equal pieces. In Grade 7 students begin to see this as a possibly infinite process. The number line is marked off into tenths, each of which is marked off into 10 hundreths, each of which is marked off into 10 thousandths, and so on ad infinitum. These finer and finer partitions constitute a sort of address system for numbers on the number line: 0.635 is, first, in the neighborhood betweeen 0.6 and 0.7, then in part of that neighborhood between 0.63 and 0.64, then exactly at 0.635.

The finite decimals are the rational numbers that eventually come to fall exactly on one of the tick marks in this decimal address system. But there are numbers that never come to rest, no

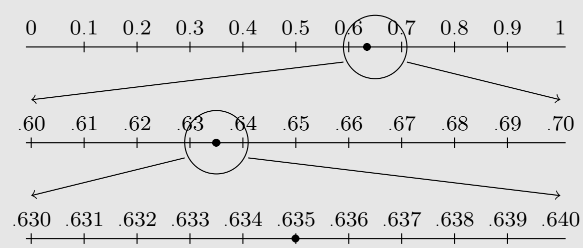
7.NS.2b Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

b Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real-world contexts.

c Apply properties of operations as strategies to multiply and divide rational numbers.

7.NS.3 Solve real-world and mathematical problems involving the four operations with rational numbers.

Zooming in on 0.635



The finite decimal 0.635 *can eventually be found sitting one of the tick marks at the thousandths level.*

matter how far down you go. For example, $\frac{1}{3}$ is always sitting one third of the way along the third subdivision. It is 0.33 plus one-third of a thousandth, and 0.333 plus one-third of a ten thousandth, and so on. The decimals 0.33, 0.333, 0.3333 are successively closer and closer approximations to $\frac{1}{3}$. We summarize this situation by saying that $\frac{1}{3}$ has an infinite decimal expansion consisting entirely of 3s

$$\frac{1}{3} = 0.3333\ldots = 0.\overline{3},$$

where the bar over the 3 indicates that it repeats indefinitely. Although a rigorous treatment of this mysterious infinite expansion is not available in middle school, students in Grade 7 start to develop an intuitive understanding of decimals as a (possibly) infinite address system through simple examples such as this.[7.NS.2d]

For those rational numbers that have finite decimal expansions, students can find those expansions using long division. Saying that a rational number has a finite decimal expansion is the same as saying that it can be expressed as a fraction whose numerator is a base-ten unit (10, 100, 1000, etc.). So if $\frac{a}{b}$ is a fraction with a finite expansion, then

$$\frac{a}{b} = \frac{n}{10} \quad \text{or} \quad \frac{n}{100} \quad \text{or} \quad \frac{n}{1000} \quad \text{or} \quad \ldots,$$

for some whole number n. If this is the case, then

$$\frac{10a}{b} = n \quad \text{or} \quad \frac{100a}{b} = n \quad \text{or} \quad \frac{1000a}{b} = n \quad \text{or} \quad \ldots.$$

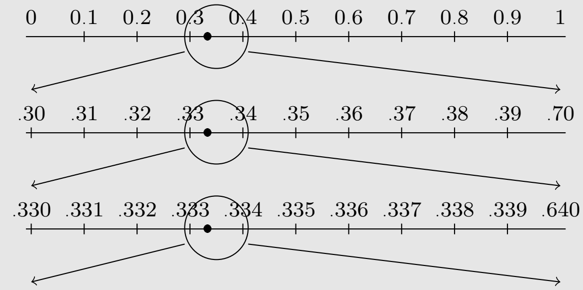
So we can find the whole number n by dividing b successively into $10a$, $100a$, $1000a$, and so on until there is no remainder.[7.NS.2d] The margin illustrates this process for $\frac{3}{8}$, where we find that there is no remainder for the division into 3000, so

$$3000 = 8 \times 375,$$

which means that

$$\frac{3}{8} = \frac{375}{1000} = 0.375.$$

Zooming in on $\frac{1}{3}$



The fraction $\frac{1}{3}$ is never eventually on one of the tick marks. It is always one third the way along the third subdivision.

7.NS.2d Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

d Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Division of 8 into 3 times a base-ten unit

$$\begin{array}{r} 3 \\ 8\,\overline{)\,30} \\ \underline{24} \\ 6 \end{array} \qquad \begin{array}{r} 37 \\ 8\,\overline{)\,300} \\ \underline{240} \\ 60 \\ \underline{56} \\ 4 \end{array} \qquad \begin{array}{r} 375 \\ 8\,\overline{)\,3000} \\ \underline{2400} \\ 600 \\ \underline{560} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

Notice that it is not really necessary to restart the division for each new base-ten unit, since the steps from the previous calculation carry over to the next one.

Grade 8

Know that there are numbers that are not rational, and approximate them by rational numbers In Grade 7 students encountered infinitely repeating decimals, such as $\frac{1}{3} = 0.\overline{3}$. In Grade 8 they understand why this phenomenon occurs, a good exercise in expressing regularity in repeated reasoning (MP8).[8.NS.1] Taking the case of $\frac{1}{3}$, for example, we can try to express it as a finite decimal using the same process we used for $\frac{3}{8}$ in Grade 7. We successively divide 3 into 10, 100, 1000, hoping to find a point at which the remainder is zero. But this never happens; there is always a remainder of 1. After a few tries it is clear that the long division will always go the same way, because the individual steps always work the same way: the next digit in the quotient is always 3 resulting in a reduction of the dividend from one base-unit to the next smaller one (see margin). Once we have seen this regularity, we see that $\frac{1}{3}$ can never be a whole number of decimal base-ten units, no matter how small they are.

A similar investigation with other fractions leads to the insight that there must always eventually be a repeating pattern, because there are only so many ways a step in the algorithm can go. For example, considering the possibility that $\frac{2}{7}$ might be a finite decimal with, we try dividing 7 into 20, 200, 2000, etc., hoping to find a point where the remainder is zero. But something happens when we get to dividing 7 into 2,000,000, the left-most division in the margin. We find ourselves with a remainder of 2. Since we started with a numerator of 2, the algorithm is going to start repeating the sequence of digits from this point on. So we are never going to get a remainder of 0. All is not in vain, however. Each calculation gives us a decimal approximation of $\frac{2}{7}$. For example, the left-most calculation in the margin tells us that

$$\frac{2}{7} = \frac{1}{1000000}\frac{2000000}{7} = 0.285714 + \frac{2}{7} \times 0.0000001,$$

and the next two show that

$$\begin{aligned}\frac{2}{7} &= 0.2857142 + \frac{6}{7} \times 0.00000001 \\ \frac{2}{7} &= 0.28571428 + \frac{4}{7} \times 0.000000001.\end{aligned}$$

Noticing the emergence of the repeating pattern 285714 in the digits, we say that

$$\frac{2}{7} = 0.\overline{285714}.$$

The possibility of infinite repeating decimals raises the possibility of infinite decimals that do not ever repeat. From the point of view of the decimal address system, there is no reason why such number should not correspond to a point on the number line. For

8.NS.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Division of 3 into 100, 1000, and 10,000

$$\begin{array}{r} 33 \\ 3\overline{)100} \\ \underline{90} \\ 10 \\ \underline{9} \\ 1 \end{array} \qquad \begin{array}{r} 333 \\ 3\overline{)1000} \\ \underline{900} \\ 100 \\ \underline{90} \\ 10 \\ \underline{9} \\ 1 \end{array} \qquad \begin{array}{r} 3333 \\ 3\overline{)10000} \\ \underline{9000} \\ 1000 \\ \underline{900} \\ 100 \\ \underline{90} \\ 10 \\ \underline{9} \\ 1 \end{array}$$

Repeated division of 3 into larger and larger base ten units shows the same pattern.

Division of 7 into multiples of 2 times larger and larger base-ten units

$$\begin{array}{r} 285714 \\ 7\overline{)2000000} \\ \underline{1400000} \\ 600000 \\ \underline{560000} \\ 40000 \\ \underline{35000} \\ 5000 \\ \underline{4900} \\ 100 \\ \underline{70} \\ 30 \\ \underline{28} \\ 2 \end{array} \qquad \begin{array}{r} 2857142 \\ 7\overline{)20000000} \\ \underline{14000000} \\ 6000000 \\ \underline{5600000} \\ 400000 \\ \underline{350000} \\ 50000 \\ \underline{49000} \\ 1000 \\ \underline{700} \\ 300 \\ \underline{280} \\ 20 \\ \underline{14} \\ 6 \end{array} \qquad \begin{array}{r} 28571428 \\ 7\overline{)200000000} \\ \underline{140000000} \\ 60000000 \\ \underline{56000000} \\ 4000000 \\ \underline{3500000} \\ 500000 \\ \underline{490000} \\ 10000 \\ \underline{7000} \\ 3000 \\ \underline{2800} \\ 200 \\ \underline{140} \\ 60 \\ \underline{56} \\ 4 \end{array}$$

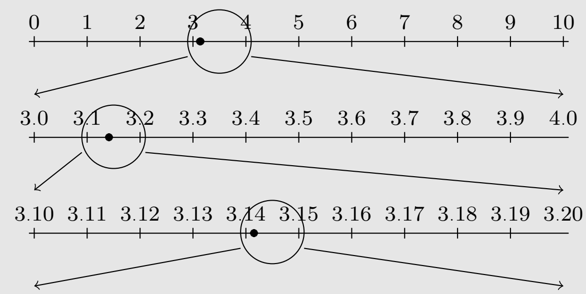
The remainder at each step is always a single digit multiple of a base-ten unit so eventually the algorithm has to cycle back to the same situation as some earlier step. From then on the algorithm produces the same sequence of digits as from the earlier step, ad infinitum.

example, the number π lives between 3 and 4, and between 3.1 and 3.2, and between 3.14 and 3.15, and so on, with each successive decimal digit narrowing its possible location by a factor of 10.

Numbers like π, which do not have a repeating decimal expansion and therefore are not rational numbers, are called *irrational*.[8.NS.1] Although we can calculate the decimal expansion of π to any desired accuracy, we cannot describe the entire expansion because it is infinitely long, and because there is no pattern (as far as we know). However, because of the way in which the decimal address system narrows down the interval in which a number lives, we can use the first few digits of the decimal expansion to come up with good decimal approximations of π, or any other irrational number. For example, the fact that π is between 3 and 4 tells us that π^2 is between 9 and 16; the fact that π is between 3.1 and 3.2 tells us that π^2 is between 9.6 and 10.3, and so on.[8.NS.2]

8.NS.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Zooming in on π



The number π has an infinite non-repeating decimal expansion which determines each successive sub-interval to zoom in on.

8.NS.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2).

High School, Number*

The Real Number System

Extend the properties of exponents to rational exponents In Grades 6–8 students began to widen the possible types of number they can conceptualize on the number line. In Grade 8 they glimpse the existence of irrational numbers such as $\sqrt{2}$. In high school, they start a systematic study of functions that can take on irrational values, such as exponential, logarithmic, and power functions. The first step in this direction is the understanding of numerical expressions in which the exponent is not a whole number. Functions such as $f(x) = x^2$, or more generally polynomial functions, have the property that if the input x is a rational number, then so is the output. This is because their output values are computed by basic arithmetic operations on their input values. But a function such as $f(x) = \sqrt{x}$ does not necessarily have rational output values for every rational input value. For example, $f(2) = \sqrt{2}$ is irrational even though 2 is rational.

The study of such functions brings with it a need for an extended understanding of the meaning of an exponent. Exponent notation is a remarkable success story in the expansion of mathematical ideas. It is not obvious at first that a number such as $\sqrt{2}$ can be represented as a power of 2. But reflecting that

$$\left(\sqrt{2}\right)^2 = 2$$

and thinking about the properties of exponents, it is natural to define

$$2^{\frac{1}{2}} = \sqrt{2}$$

since if we follow the rule $(a^b)^c = a^{bc}$ then

$$\left(2^{\frac{1}{2}}\right)^2 = 2^{\frac{1}{2}\cdot 2} = 2^1 = 2.$$

Similar reasoning leads to a general definition of the meaning of a^b whenever a and b are rational numbers.[N-RN.1] It should be noted high school mathematics does not develop the mathematical ideas necessary to prove that numbers such as $\sqrt{2}$ and $3^{\frac{1}{5}}$ actually exist;

N-RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

*This progression concerns Number and Quantity standards related to number. The remaining standards are discussed in the Quantity Progression.

in fact all of high school mathematics depends on the fundamental assumption that properties of rational numbers extend to irrational numbers. This is not unreasonable, since the number line is populated densely with rational numbers, and a conception of number as a point on the number line gives reassurance from intuitions about measurement that irrational numbers are not going to behave in a strangely different way from rational numbers.

Because rational exponents have been introduced in such a way as to preserve the laws of exponents, students can now use those laws in a wider variety of situations. For example, they can rewrite the formula for the volume of a sphere of radius r,

$$V = \frac{4}{3}\pi r^3,$$

to express the radius in terms of the volume,[N-RN.2]

$$r = \left(\frac{3}{4}\frac{V}{\pi}\right)^{\frac{1}{3}}.$$

N-RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers An important difference between rational and irrational numbers is that rational numbers form a number system. If you add, subtract, multiply, or divide two rational numbers, you get another rational number (provided the divisor is not 0 in the last case). The same is not true of irrational numbers. For example, if you multiply the irrational number $\sqrt{2}$ by itself, you get the rational number 2. Irrational numbers are defined by not being rational, and this definition can be exploited to generate many examples of irrational numbers from just a few.[N-RN.3] For example, because $\sqrt{2}$ is irrational it follows that $3+\sqrt{2}$ and $5\sqrt{2}$ are also irrational. Indeed, if $3+\sqrt{2}$ were an irrational number, call it x, say, then from $3+\sqrt{2} = x$ we would deduce $\sqrt{2} = x-3$. This would imply $\sqrt{2}$ is rational, since it is obtained by subtracting the rational number 3 from the rational number x. But it is not rational, so neither is $3+\sqrt{2}$.

N-RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Although in applications of mathematics the distinction between rational and irrational numbers is irrelevant, since we always deal with finite decimal approximations (and therefore with rational numbers), thinking about the properties of rational and irrational numbers is good practice for mathematical reasoning habits such as constructing viable arguments and attending to precisions (MP.3, MP.6).

Complex Numbers

That complex numbers have a practical application is surprising to many. But it turns out that many phenomena involving real numbers become simpler when the real numbers are viewed as a subsytem of the complex numbers. For example, complex solutions of differential equations can give a unified picture of the behavior of real solutions. Students get a glimpse of this when they study complex solutions of quadratic equations. When complex numbers are brought into the picture, every quadratic polynomial can be expressed as a product of linear factors:

$$ax^2 + bx + c = a(x - r)(x - s).$$

The roots r and s are given by the quadratic formula:

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

When students first apply the quadratic formula to quadratic equations with real coefficients, the square root is a problem if the quantity $b^2 - 4ac$ is negative. Complex numbers solve that problem by introducing a new number, i, with the property that $i^2 = -1$, which enables students to express the solutions of any quadratic equation.[N-CN.7]

N-CN.7 Solve quadratic equations with real coefficients that have complex solutions.

One remarkable fact about introducing the number i is that it works: the set of numbers of the form $a + bi$, where $i^2 = -1$ and a and b are real numbers, forms a number system. That is, you can add, subtract, multiply and divide two numbers of this form and get another number of the same form as the result. We call this the system of complex numbers.[N-CN.1]

N-CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

All you need to perform operations on complex numbers is the fact that $i^2 = -1$ and the properties of operations.[N-CN.2] For example, to add $3 + 2i$ and $-1 + 4i$ we write

N-CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

$$(3 + 2i) + (-1 + 4i) = (3 + -1) + (2i + 4i) = 2 + 6i,$$

using the associative and commutative properties of addition, and the distributive property to pull the i out, resulting in another complex number. Multiplication requires using the fact that $i^2 = -1$:

$$(3 + 2i)(-1 + 4i) = -3 + 10i + 8i^2 = -3 + 10i - 8 = -11 + 10i.$$

Division of complex numbers is a little tricker, but with the discovery of the complex conjugate $a - bi$ we find that every non-zero complex number has a multiplicative inverse.[N-CN.3] If at least one of a and b is not zero, then

N-CN.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

$$(a + bi)^{-1} = \frac{1}{a^2 + b^2}(a - bi)$$

because

$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

Students who continue to study geometric representations of complex numbers in the complex plane use both rectangular and polar coordinates which leads to a useful geometric interpretation of the operations.[N-CN.4, N-CN.5] The restriction of these geometric interpretations to the real numbers yields and interpretation of these operations on the number line.

One of the great theorems of modern mathematics is the Fundamental Theorem of Algebra, which says that every polynomial equation has a solution in the complex numbers. To put this into perspective, recall that we formed the complex numbers by creating a solution, i, to just one special polynomial equation, $x^2 = -1$. With the addition of this one solution, it turns out that every polynomial equation, for example $x^4 + x^2 = -1$, also acquires a solution. Students have already seen this phenomenon for quadratic equations.[N-CN.9]

Although much of the study of complex numbers goes beyond the college and career ready threshold, as indicated by the (+) on many of the standards, it is a rewarding area of exploration for advanced students.

N-CN.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

N-CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.